Normal Modes of Rotating Magnetic Stars

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ABSTRACT

We investigate the effect of a magnetic field on the global oscillation modes of a rotating fluid star in the magnetohydrodynamic approximation. We present general equations for the modification of any type of fluid mode due to a general magnetic field which is not aligned with the star's spin axis. In the case of any internal dipole magnetic field we derive the equations for the frequency corrections to the r-modes. We solve for the frequency correction explicitly for the case when the internal dipole field is force-free, including the uniform density case. In the weak-field limit, the spatial form of the r-mode velocity perturbation is unchanged, but the magnitude of the frequency in the rotating frame increases.

Subject headings: stars: magnetic – stars: rotation – stars: oscillations – stars: neutron

1. Introduction

The observation of oscillations of the Sun and other stars has led to great advances in our understanding of the internal structure of stars. While the oscillations of non-rotating, non-magnetic stars are well understood, less is known about the oscillations of stars which are rotating and possess a magnetic field (Cox 1980; Unno et al. 1989). The main classes of stars for which magnetic fields and rotation both play important roles are the rapidly oscillating Ap (RoAp) stars, the magnetic white dwarf stars and neutron stars. The Ap stars are hydrogen burning stars with strong magnetic fields. Rapid oscillations (relative to the spin frequency) have been observed in the subclass of stars known as RoAp stars (Kurtz 1990). These oscillations have been identified as high overtone p-modes. In addition, evidence for line splitting by rotation has been found. These stars are modeled with a magnetic field tilted with respect to the spin axis of the star.

Numerous white dwarf stars have been observed to oscillate (Gautschy & Saio 1996). The oscillations of the white dwarfs have been identified as g-modes. So far no magnetic field larger than $\sim kG$ (Schmidt & Grauer 1997) has been observed for any of the pulsating white dwarfs. On the other hand, a class of white dwarfs with magnetic fields up to $10^9 G$ (Wickramasinghe & Ferrario 2000) known as magnetic white dwarfs (MWD) has shown no evidence of pulsations. The lack of pulsations in the MWDs may be due to magnetic suppression of pulsations or to pulsation amplitudes which are not large enough to be detectable.

The observation of quasiperiodic pulsar microstructure (Boriakoff 1976; Cordes et al 1990) has led to the suggestion (Van Horn 1980) that neutron star oscillations had been observed. While the evidence for electromagnetic detection of neutron star oscillation modes is inconclusive, it may be possible that neutron star oscillations will be detectable by advanced gravitational wave detectors. It has also been proposed that the soft gamma repeaters (SGR) are high magnetic field neutron stars with fields as high as $10^{15}G$ (Thompson & Duncan 1995). This hypothesis is supported by observations of objects such as SGR 1900+14 which appears to have a magnetic dipole field strength in the range $\sim 2 - 8 \times 10^{14}G$ (Kouveliotou 1998). Torsion modes in the crusts of these neutron stars may have been detected in SGR outbursts (Duncan 1998).

The relative importance of magnetic fields and rotation in different types of stellar systems varies by many orders of magnitude, as can be seen in Table 1, where the ratio of magnetic field energy, $\mathcal{M} = B_{in}^2 R^3/6$ to rotational kinetic energy, $T = MR^2(2\pi/P)^2/5$, is given for typical stars. In Table 1 we give values for the average magnetic field, and the magnetic field energy is calculated as though the star's magnetic field is uniform. For millisecond pulsars and typical white dwarf stars, rotation is much more important than magnetic fields at global scales. At the other end of the scale, if the soft gamma repeaters can be modeled as high magnetic field neutron stars, magnetic fields are dominant over rotation. The magnetic white dwarf stars and some high magnetic field neutron stars have both magnetic field and rotation energies of similar sizes. Clearly any mathematical method which treats the ratio of magnetic field energy to rotational energy as a small perturbation can not describe the phenomenology of all types of stars listed in Table 1.

In this paper we introduce a formalism for computing global oscillation modes of magnetic, rotating Newtonian fluid stars. Our method makes use of a slow-rotation expansion, but all other aspects are nonperturbative, in that the modes of stars with any value of the ratio of $\mathcal{M}/T$ can be computed. The method of computation is particularly useful for the computation of magnetically modified r-modes and inertial modes which vanish in the limit of zero rotation. For illustrative purposes the magnetic corrections to the r-modes of a constant density star with a homogeneous magnetic field inclined at arbitrary angle to the star's

spin axis will be presented. In the example presented, the computed modes are only valid in the limit of small $\mathcal{M}/T$. The full treatment of the modes of stars with strong magnetic fields is more involved than our example and will be presented elsewhere. It is straight-forward to extend these calculations to stars with realistic equations of state and more complicated magnetic fields.

There is already an extensive literature on the pulsations of magnetic stars. Radial oscillations of nonrotating magnetic stars were first investigated by Chandrasekhar & Fermi (1953). Ledoux & Simon (1957) computed the magnetic corrections to the nonradial f-modes of a nonrotating star. In these early calculations, the overall effect of the magnetic field was assumed to be weak, in the sense that the energy stored in the magnetic field is small compared to the star's gravitational binding energy. While this is generally true for all stars of interest, the magnetic pressure can dominate over gas pressure near the star's surface. This suggests that a perturbative approach will break down near the surface of a star. Carroll et al. (1986) introduced a cylindrical model of the region near the magnetic pole at the surface of a neutron star and computed the spectrum of nonradial oscillations of the model problem. A magnetic boundary layer approach was adopted by Dziembowski & Goode (1996) and Bigot et al. (2000) as an alternative method for finding the magnetically modified modes near the stellar surface.

Computations of nonradial oscillations of rotating magnetic stars are more complicated than the calculation of modes of magnetic nonrotating stars. One approach is perturbative: the star's rotation and magnetic field can be taken to be small corrections to the modes of nonrotating, nonmagnetic stars. This perturbation approach has been applied to the calculation of magnetically modified pressure modes of rotating stars (Shibahashi & Takata 1993; Takata & Shibahashi 1994, 1995). The work on modes of rotating magnetic stars has so far focused on the pressure modes. However, in the absence of a magnetic field a rotating star has another family of modes, the inertial modes, which are driven by the Coriolis force. Malkus (1967) computed the magnetic generalization of these modes for the case of a star whose magnetic field is generated by an electric current which is parallel to the fluid's angular velocity. In this special case the equations describing normal modes can be separated in oblate spheroidal coordinates. It is of interest, however, to understand the magnetic generalizations of the inertial modes for magnetic fields which are tilted with respect to the star's rotation axis, as is the situation common in astrophysics. If excited in the Ap stars, these modes may create beat frequencies with the high order p-modes which may be observable. (The inertial modes have a very small radial component to their oscillation and are unlikely to provide an observable luminosity variation on their own.) Another motivation for computing the magnetically modified inertial modes is due to the prediction that it may be possible for a subset of the inertial modes, the r-modes, to be driven

unstable by gravitational radiation (Andersson 1998; Friedman & Morsink 1998; Lindblom et al. 1998) in young neutron stars. In a series of papers Rezzolla et al. (2000, 2001a,b) have examined the consequences of a large amplitude r-mode and shown that it would have the effect of amplifying the star's magnetic field and hindering the further growth of the mode. This effect could play an important role in the early evolution of a neutron star's spin and magnetic field. Further work by Ho & Lai (2000) provided further insight on the effect of a growing r-mode on the magnetic field and calculated the form of magnetically modified r-modes in a plane-wave approximation. One of the motivations for the present paper is to provide a framework for computing the form of the magnetically modified global r-modes. The formalism and calculations presented in this paper are only for global oscillation modes of an infinitely conducting fluid. We have not included features such as a crust or a magnetosphere which would exist in the case of neutron stars. It should be possible to extend the present formalism to include interactions with crustal oscillation modes and a magnetosphere.

The structure of this paper is as follows. In section 2 we review the equations describing the modes of rotating magnetic fluid stars. The form of the equilibrium magnetic field is discussed in section 3. In section 4 we show how the frequencies of the r-modes are modified by the presence of a dipole magnetic field. We examine two model interior fields, the uniform magnetic field and the force-free magnetic field. In the case of a uniform magnetic field the frequency in the rotating frame of an r-mode with quantum number ℓ is

$$\sigma = -\frac{2\Omega}{\ell+1}\left(1 - \frac{\mathcal{M}}{T}\frac{1}{10}(\ell+1)(2\ell+3)\left[1 + \frac{1}{2}(\ell(\ell+1)-3)\sin^2\alpha\right]\right), \qquad (1)$$

where Ω is the angular velocity of the star, $\mathcal{M}$ is the energy stored in the magnetic field, T is the star's rotational kinetic energy and α is the angle between the magnetic field's axis and the spin axis. In the case of the force-free magnetic field, numerical results are presented. Finally, we conclude in section 5 with remarks about further problems which can be attacked with the present formalism.

2. Normal Modes of Rotating Stars with Magnetic Fields

In this paper we study the magneto-hydrodynamic (MHD) perturbations of a uniformly rotating star endowed with a magnetic field. As we do not assume that the symmetry axis of the magnetic field is aligned with the star's rotation axis, magnetic multipole radiation is emitted, and no equilibrium solution exists. However, we only consider timescales short compared to the characteristic stellar spin-down time due to the nonaxisymmetric magnetic field. In this sense an equilibrium solution exists. We work within the ideal MHD framework

(Jackson 1975) where the star's fluid is assumed to be electrically neutral and to have infinite conductivity. With the assumption of infinite conductivity, the magnetic field lines are frozen into the fluid, and the magnetic field pattern rotates at the same rate as the star.

The equilibrium star is described by the density ρ, fluid pressure p, gravitational potential ϕ, velocity $\mathbf{v}$ and magnetic field $\mathbf{B}$. Eulerian perturbations of these quantities are denoted with the symbol δ. For infinite conductivity, the Eulerian perturbations of the magnetic field and current density are

$$\frac{\partial}{\partial t}\delta\mathbf{B} \;=\; \boldsymbol{\nabla}\times(\delta\mathbf{v}\times\mathbf{B}) \tag{2a}$$

$$\delta\mathbf{E} \;=\; -\frac{1}{c}\delta\mathbf{v}\times\mathbf{B} \tag{2b}$$

$$\frac{1}{c}\delta\mathbf{J} \;=\; \frac{1}{4\pi}\left(\boldsymbol{\nabla}\times\delta\mathbf{B}-\frac{1}{c}\frac{\partial}{\partial t}\delta\mathbf{E}\right), \tag{2c}$$

where $\delta\mathbf{v}=\partial_t\boldsymbol{\xi}$ and the perturbed density, pressure and gravitational field are given in standard works on perturbation theory such as Unno et al. (1989). The fluid displacement vector $\boldsymbol{\xi}$ is a solution of Euler's equation,

$$\ddot{\boldsymbol{\xi}}+2\mathcal{C}\dot{\boldsymbol{\xi}}+\mathcal{W}\boldsymbol{\xi}=\frac{1}{\rho}\mathbf{F}^{mag}, \tag{3}$$

where the operators $\mathcal{C}$ and $\mathcal{W}$ are defined by

$$\mathcal{C}\boldsymbol{\xi}\equiv\Omega\times\boldsymbol{\xi}, \quad \mathcal{W}\boldsymbol{\xi}\equiv\frac{1}{\rho}\boldsymbol{\nabla}\delta p-\frac{\boldsymbol{\nabla}p}{\rho^2}\delta\rho+\boldsymbol{\nabla}\delta\phi, \tag{4}$$

and the magnetic force is

$$\mathbf{F}^{mag}=\frac{1}{c}\left(\delta\mathbf{J}\times\mathbf{B}+\mathbf{J}\times\delta\mathbf{B}\right). \tag{5}$$

The possible perturbations of a magnetized rotating star are found by specifying the equilibrium configuration of the star and then solving equation (3). This is a nontrivial problem, and the rest of this paper will be concerned with a method for finding normal mode solutions of Euler's equation.

In this paper we adopt the method introduced by Schenk et al. (2002) where the perturbations are expanded in the spatial eigenfunctions of Euler's equation with zero magnetic field. The normal modes of rotating stars with no magnetic fields (or any other external forces) are solutions of Euler's equation (3) with $F^{mag}=0$ of the form $e^{-i\omega_A t}\xi_A(x)$ where the spatial eigenfunction $\xi_A(x)$ satisfies the equation

$$-\omega_A^2\boldsymbol{\xi}_A-2i\omega_A\mathcal{C}\boldsymbol{\xi}_A+\mathcal{W}\boldsymbol{\xi}_A=0. \tag{6}$$

Upper-case Latin subscripts are used to label solutions and correspond to the unique set of quantum numbers which describe the solution. We will make use of the inner product defined by

$$< \boldsymbol{\xi}_A, \mathcal{O}\boldsymbol{\xi}_B >= \int d^3 x \rho(x) \boldsymbol{\xi}_A^* \cdot \mathcal{O}\boldsymbol{\xi}_B, \tag{7}$$

for any operator $\mathcal{O}$. An important property of the modes of rotating stars discussed by Schenk et al. (2002) is that the modes are not orthogonal with respect to the usual inner product. Instead, the modes obey the relation

$$2\epsilon_A \delta_{AB} = \omega_A(\omega_A + \omega_B) < \boldsymbol{\xi}_B, \boldsymbol{\xi}_A > +2\omega_A < \boldsymbol{\xi}_B, i\mathcal{C}\boldsymbol{\xi}_A >= \omega_A\omega_B < \boldsymbol{\xi}_B, \boldsymbol{\xi}_A > +\omega_A < \boldsymbol{\xi}_B, \mathcal{W}\boldsymbol{\xi}_A > \tag{8}$$

which defines ϵ_A, the rotating frame energy of the mode (Friedman & Schutz 1978a).

The lack of a simple orthogonality relation for the modes of rotating stars has the following consequence. If a general perturbation is expanded into a sum over modes of the form $\boldsymbol{\xi}(x, t) = \sum_B a_B(t)\boldsymbol{\xi}_B(x)$, the expansion coefficients are not given by the usual formula $a_A(t) \propto < \boldsymbol{\xi}_A, \boldsymbol{\xi}(x, t) >$ as would be true in the case of zero rotation. An alternative phase space expansion has been introduced by Schenk et al. (2002) to circumvent this complication. In the phase space approach, the perturbation and its first time derivative have expansions of the form

$$\boldsymbol{\xi}(x, t) = \sum_B q_B(t)\boldsymbol{\xi}_B(x), \quad \text{and} \quad \dot{\boldsymbol{\xi}}(x, t) = -i\sum_B \omega_B q_B(t)\boldsymbol{\xi}_B(x), \tag{9}$$

where the expansion coefficients are given by (Schenk et al. 2002)

$$q_A(t) = \frac{\omega_A}{2\epsilon_A} \left\langle \boldsymbol{\xi}_A, \omega_A \boldsymbol{\xi}(\mathbf{r}, t) + i\dot{\boldsymbol{\xi}}(\mathbf{r}, t) + 2i\mathcal{C}\boldsymbol{\xi}(\mathbf{r}, t) \right\rangle. \tag{10}$$

The equations of motion for the expansion coefficients take on the simple form derived by Schenk et al. (2002)

$$\dot{q}_A(t) + i\omega_A q_A(t) = \frac{i\omega_A}{2\epsilon_A} \left\langle \boldsymbol{\xi}_A, \frac{1}{\rho}\mathbf{F}^{mag}[\boldsymbol{\xi}(\mathbf{r}, t)] \right\rangle = \frac{i\omega_A}{2\epsilon_A} \sum_B q_B \left\langle \boldsymbol{\xi}_A, \frac{1}{\rho}\mathbf{F}_B[\boldsymbol{\xi}_B(\mathbf{r})] \right\rangle, \tag{11}$$

where the magnetic force (5) has an expansion of the form $F^{mag} = \sum_A q_A(t) F_A[\boldsymbol{\xi}_A(\mathbf{r})]$.

The term $\mathbf{F}_A$ is the Lorentz force created by the fluid motion $\boldsymbol{\xi}_A$. We now define dimensionless magnetic coupling coefficients through

$$\kappa_{AB} = \frac{1}{\mathcal{M}} < \boldsymbol{\xi}_A, \frac{1}{\rho}\mathbf{F}_B > . \tag{12}$$

The integral $< \boldsymbol{\xi}_A, \frac{1}{\rho}\mathbf{F}_B >$ is the work done by the force $\mathbf{F}_B$ when the fluid is displaced through the distance $\boldsymbol{\xi}_A$. The coefficients κ_{AB} can then be thought of as the ratio of work done by the perturbed Lorentz force to the total magnetic energy stored in the equilibrium star. The diagonal entries of the matrix κ correspond to the work done against the fluid motion $\boldsymbol{\xi}_A$ which generates the Lorentz force $\mathbf{F}_A$. Since the magnetic forces will tend to oppose the motion of the fluid, we expect that the diagonal entries in the matrix of coupling coefficients will be negative.

After integrating by parts, the coupling coefficients take on the form

$$\kappa_{AD} = -\frac{1}{4\pi\mathcal{M}} \int \left(\kappa_{AD}^{(1)}(r) - \kappa_{AD}^{(2)}(r) - \frac{\omega_D^2}{c^2}\kappa_{AD}^{(3)}(r) \right) r^2 dr \tag{13}$$

$$\kappa_{AD}^{(1)}(r) = \int \boldsymbol{\nabla} \times (\boldsymbol{\xi}_A^* \times \mathbf{B}) \cdot \boldsymbol{\nabla} \times (\boldsymbol{\xi}_D \times \mathbf{B}) \, d\Omega, \tag{13a}$$

$$\kappa_{AD}^{(2)}(r) = \int (\boldsymbol{\xi}_A^* \times (\boldsymbol{\nabla} \times \mathbf{B})) \cdot \boldsymbol{\nabla} \times (\boldsymbol{\xi}_D \times \mathbf{B}) \, d\Omega \tag{13b}$$

$$\kappa_{AD}^{(3)}(r) = \int (\boldsymbol{\xi}_A^* \times \mathbf{B}) \cdot (\boldsymbol{\xi}_D \times \mathbf{B}) \, d\Omega. \tag{13c}$$

The diagonal elements $\kappa_{AA}^{(1)}$ and $\kappa_{AA}^{(3)}$ are positive definite. In most cases $\omega R \ll c$ so that the term proportional to $\kappa_{AD}^{(3)}$ can be neglected. This last term is large typically only for neutron stars. However, since we are using Newtonian gravity, the errors incurred in neglecting $\kappa_{AD}^{(3)}$ are similar to the errors coming from the neglect of general relativity. Although most authors neglect the term $\kappa_{AD}^{(3)}$, we have chosen to keep it explicitly in our equations for the sake of completeness.

Normal mode solutions to the MHD perturbation equations can be found by making the ansatz

$$q_A(t) = c_A \exp(-i\sigma t) \tag{14}$$

where the c_A are constants and σ is the mode frequency in the rotating frame. Substituting the ansatz (14) into the equations of motion, we reduce the equations to the form

$$\omega_A \left(c_A - \frac{\mathcal{M}}{2\epsilon_A} \sum_D \kappa_{AD} c_D \right) = \sigma c_A, \tag{15}$$

which is an eigenvalue problem for the frequency σ and eigenvector components c_A. In order to solve the eigenvalue problem, we must truncate the system to a finite number of modes. Suppose, for instance, that we chose to truncate the basis to a set of N eigenmodes of the non-magnetic system. Solving the system of equations (15) will result in N eigenvectors and eigenfrequencies of the magnetic system. The dimension of the vector space can then be

increased by one by adding one more non-magnetic basis vector and solving for the N+1 magnetic modes. The difference between the original N frequencies can then be calculated and the process iterated until the frequency differences between successive iterations is suitably small. This procedure would be difficult to implement if it happened that all of the modes were coupled to each other. However, in the case of dipole magnetic fields, it will be shown in section 4 that the angular momentum selection rules restrict the possible mode couplings so that each mode couples to only a small number of other modes. As a result, most of the κ_{AD} coupling coefficients vanish. This result will continue to hold for magnetic fields of higher mulitpoles.

The constants ϵ_A appearing in the eigenvalue equation (15) are the rotating frame energies of the modes of a nonmagnetic star. These energies depend on the normalization chosen for the modes. Since the problem of finding modes of a magnetized star is linear, the final solution corresponding to an eigenfrequency σ and eigenvector components c_A is independent of the normalizations chosen for the spatial eigenfunctions of the nonmagnetized star. This allows us to choose any convenient normalization scheme which simplifies the form of the eigenvalue equation. A natural choice of normalization results by noting that the rotating frame energy is proportional to the square of the nonmagnetized mode frequency ω_A. The mode frequencies of rotating nonmagetized stars fall into two classes, frequencies which in the nonrotating limit are finite (such as f-modes, p-modes and g-modes) and frequencies which vanish in the nonrotating limit (inertial modes, such as the r-modes). A simple scheme is to choose the normalization of all modes with finite frequency in the nonrotating limit so that their energy is $\epsilon_A = |W|$ where W is the gravitational potential energy of a constant density star with the same mass and radius as the star under study. Similarly, a suitable normalization is to choose the energy of the inertial modes to be $\epsilon_A = T$ where T is the rotational kinetic energy of the equivalent constant density star. This choice of normalization suggests the following approximation scheme for weak magnetic fields. When $\mathcal{M}/T \ll 1$ and $\mathcal{M}/|W| \ll 1$ the off-diagonal entries in the matrix

$$N_{AD} = \delta_{AD} - \frac{\mathcal{M}}{2\epsilon_A}\kappa_{AD} \tag{16}$$

will be much smaller than the diagonal entries. When N_{AD} can be approximated as diagonal, $N_{AD} \simeq \delta_{AD}\left(1 - \frac{\mathcal{M}}{2\epsilon_A}\kappa_{AA}\right)$ the eigenvalue problem is greatly simplified, yielding spatial eigenfunctions identical to the eigenfunctions of the nonmagnetic star. In the case of modes which vanish in the non-rotating limit, such as the r-modes, only one mode exists for each spatial eigenfunction. In this case, when matrix N_{AD} can be approximated by a diagonal matrix, the equation for the MHD frequency is simply given by

$$\sigma = \omega_A\left(1 - \frac{\mathcal{M}\kappa_{AA}}{2T}\right). \tag{17}$$

If the diagonal terms in the coupling matrix κ are purely real, as in the case of a force-free magnetic field, the resulting frequency σ will also be real. In the case of r-modes of stars with force-free magnetic fields, the term $\kappa_{AA}^{(2)}(r)$ vanishes and the diagonal entries κ_{AA} are all negative definite. As a result, the magnetic field increases the absolute value of the r-mode frequencies.

Implicit in this method is the assumption that the spatial eigenfunctions of a rotating star form a complete basis. However, it has been shown by Dyson & Schutz (1979) that the normal mode of a rotating star are complete only in a weak sense. Dyson & Schutz (1979) have shown that when the frequencies of the normal modes are all real, there are no exponentially growing solutions of the initial-value problem. In the case of purely real frequencies it is then possible to express perturbations in terms of normal modes. Dyson & Schutz (1979) have stressed that their result allows the approximation of an infinite-dimensional problem by a finite-dimensional problem. These results on completenesss are only valid when the eigenfunctions are square integrable. In some special cases, such as an incompressible perfect fluid confined to a spherical shell, it has been shown (Rieutord et al. 2001) that there exists perturbations which are not square integrable and that only the r-modes are regular solutions. For the case of a spherical shell, an expansion in normal modes will not represent all perturbations.

Clearly, this method relies heavily on the computation of the magnetic coupling coefficients κ_{AD}. This is not a simple task. In the following sections we discuss selection rules and strategies for computing the coupling coefficients.

2.1.   Spin-weighted Mode Decomposition

The calculation of the κ_{AD} coupling coefficients is nontrivial and requires a systematic strategy. In our computation we make use of spin-weighted spherical harmonics and the formalism introduced by Schenk et al. (2002).

To begin with, we introduce a complex basis $\{\boldsymbol{e}_0, \boldsymbol{e}_+, \boldsymbol{e}_-\}$ related to the usual orthonormal basis $\{\boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\}$ by

$$\boldsymbol{e}_0 = \boldsymbol{e}_{\hat{r}}, \quad \boldsymbol{e}_{\pm} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_{\hat{\theta}} \pm i\boldsymbol{e}_{\hat{\phi}}\right). \tag{18}$$

These unit vectors have the following scalar products

$$\boldsymbol{e}_0 \cdot \boldsymbol{e}_0 = \boldsymbol{e}_+ \cdot \boldsymbol{e}_- = 1, \quad \boldsymbol{e}_0 \cdot \boldsymbol{e}_{\pm} = \boldsymbol{e}_+ \cdot \boldsymbol{e}_+ = \boldsymbol{e}_- \cdot \boldsymbol{e}_- = 0. \tag{19}$$

Any vector $\boldsymbol{\xi}$ has an expansion of the form

$$\boldsymbol{\xi} = \xi_0 \boldsymbol{e}_0 + \xi_- \boldsymbol{e}_+ + \xi_+ \boldsymbol{e}_- \tag{20a}$$

$$\xi_0 = \xi_r, \quad \xi_\pm = \boldsymbol{\xi} \cdot \boldsymbol{e}_\pm = \frac{1}{\sqrt{2}} \left(\xi_\theta \pm i\xi_\phi \right), \tag{20b}$$

where $\xi_r, \xi_\theta, \xi_\phi$ are the components of the vector in the usual orthonormal basis.

The spherical harmonics with spin-weight s are defined by (Campbell 1971)

$$_sY_{\ell m}(\theta, \phi) = \sqrt{\frac{(2\ell+1)}{4\pi}} d^\ell_{-sm}(\theta) e^{im\phi} \tag{21}$$

where the $d^\ell_{sm}(\theta)$ are the matrix representations for rotations through an angle θ discussed in detail by Edmonds (1974). When the spin-weight $s = 0$, the spin-weighted spherical harmonics reduce to the regular spherical harmonics. From the definitions of the spin spherical harmonics, it follows that (Schenk et al. 2002)

$$\boldsymbol{e}_\pm \cdot \boldsymbol{\nabla} Y_{\ell m} = \mp \frac{1}{\sqrt{2}r} \sqrt{\ell(\ell+1)} \, _{\pm 1}Y_{\ell m} \tag{22a}$$

$$\boldsymbol{e}_\pm \cdot \mathbf{L} Y_{\ell m} = -\frac{1}{\sqrt{2}} \sqrt{\ell(\ell+1)} \, _{\pm 1}Y_{\ell m}, \tag{22b}$$

where $\mathbf{L} = -i\mathbf{r} \times \boldsymbol{\nabla}$.

The spatial eigenmodes of a rotating, unmagnetized star have azimuthal angular dependence $\exp(im_A\phi)$, but in general do not have a definite quantum number ℓ. The modes can be written in the general form (see, for example Lockitch & Friedman (1999))

$$\boldsymbol{\xi}_A(x) = \sum_{\ell=|m_A|}^\infty \left[\frac{W_{\ell m_A}(r)}{r} Y_{\ell m_A} \boldsymbol{e}_{\hat{r}} + V_{\ell m_A}(r) \boldsymbol{\nabla} Y_{\ell m_A} - \frac{U_{\ell m_A}(r)}{r} \mathbf{L} Y_{\ell m_A} \right]. \tag{23}$$

Making use of equations (22a) and (22b), the general mode expansion (23) is

$$\boldsymbol{\xi}_A(x) = \sum_{\ell'} \left(f_0^{\ell m_A}(r) \, _0Y_{\ell m_A} \boldsymbol{e}_0 + f_+^{\ell m_A}(r) \, _{+1}Y_{\ell m_A} \boldsymbol{e}_- + f_-^{\ell m_A}(r) \, _{-1}Y_{\ell m_A} \boldsymbol{e}_+ \right) \tag{24}$$

where the functions $f_s^{\ell m_A}$ are given by

$$f_0^{\ell m_A}(r) = \frac{W_{\ell m_A}}{r} \quad \text{and} \quad f_\pm^{\ell m_A}(r) = \mp \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{r} \left(V_{\ell m_A} \mp U_{\ell m_A} \right). \tag{25}$$

3. The Equilibrium Magnetic Field

A static magnetic field which is symmetric about an axis can be written in the general form

$$\mathbf{B}(r, \theta', \phi') = \sum_\ell \sqrt{\frac{4\pi}{2\ell+1}} \left(\beta_{1,\ell}(r) Y_{\ell 0}(\theta', \phi') \boldsymbol{e}_{\hat{r}} - r\beta_{2,\ell}(r) \boldsymbol{\nabla}' Y_{\ell 0}(\theta', \phi') - i\beta_{3,\ell}(r) \mathbf{L}' Y_{\ell 0}(\theta', \phi') \right), \tag{26}$$

where θ' and ϕ' are angular coordinates measured with respect to the magnetic field's axis of symmetry and $\mathbf{\nabla}'$ and $\mathbf{L}'$ are angular operators with respect to the magnetic field's symmetry axis. The expansion of the general magnetic field in the spin-weighted basis is

$$B(r, \theta', \phi') = B_0'(r, \theta', \phi')\boldsymbol{e}_0 + B_+'(r, \theta', \phi')\boldsymbol{e}_- + B_-'(r, \theta', \phi')\boldsymbol{e}_+, \tag{27}$$

where each component has the expansion in spin-weighted spherical harmonics

$$B'_s = \sum_\ell b'^\ell_s(r) \, {}_sY_{\ell 0}(\theta', \phi'), \tag{28}$$

and the set of functions $b'^\ell_s(r)$ are given by

$$b'^\ell_0 = \sqrt{\frac{4\pi}{2\ell+1}}\beta_{1,\ell} \quad \text{and} \quad b'^\ell_\pm = \pm\sqrt{\frac{4\pi}{2\ell+1}}\sqrt{\frac{\ell(\ell+1)}{2}}(\beta_{2,\ell} \pm i\beta_{3,\ell}). \tag{29}$$

In general, the axis of symmetry of the magnetic field will not coincide with the star's spin axis. Suppose that the magnetic field is tilted by an angle α from the star's spin axis. If the angles θ and ϕ are measured with respect to the spin axis, the spin-weighted spherical harmonics in the two coordinate systems are related by

$$_sY_{\ell k}(\theta', \phi') = \sum_m d^\ell_{mk}(\alpha) \, {}_sY_{\ell m}(\theta, \phi). \tag{30}$$

As a result, the magnetic field can also be written in terms of the spin-axis coordinate system

$$B(r, \theta, \phi) \;=\; B_0(r, \theta, \phi)\boldsymbol{e}_0 \;+\; B_+(r, \theta, \phi)\boldsymbol{e}_- \;+\; B_-(r, \theta, \phi)\boldsymbol{e}_+ \tag{31a}$$

$$B_s = \sum_{\ell,m} b_s^{\ell,m}(r) \, {}_sY_{\ell m}(\theta, \phi) \tag{31b}$$

$$b_s^{\ell,m}(r) = b'^\ell_s(r)d^\ell_{m0}(\alpha). \tag{31c}$$

In order for the magnetic field to be real, the coefficients must obey the relations

$$b_0^{\ell,m*}(r) \;=\; b_0^{\ell,m}(r) \tag{32a}$$

$$b_+^{\ell,m*}(r) \;=\; -b_-^{\ell,m}(r) \tag{32b}$$

where $*$ denotes complex conjugation.

The exterior magnetic field of most stars is approximately that of a dipole, which has the form $\mathbf{B}(r', \theta', \phi') = -B_p R^3 \sqrt{4\pi/3} \, \mathbf{\nabla}'(Y_{1\,0}(\theta', \phi')/r'^2)$ in the coordinate system aligned with the magnetic field's symmetry axis. In this case the exterior magnetic is described by coefficients $\beta_1(r)/2 = \beta_2(r) = B_p R^3/r^3$ and $\beta_3(r) = 0$. (We will now drop the subscript ℓ

for the β coefficients since we will only be considering the dipole case $\ell = 1$.) In the case of a dipole field, the only coefficients of the spin matrix $d^1(\alpha)$ of interest are

$$d^1_{10}(\alpha) = \frac{1}{\sqrt{2}} \sin \alpha, \quad d^1_{00}(\alpha) = \cos \alpha, \quad d^1_{-10}(\alpha) = -\frac{1}{\sqrt{2}} \sin \alpha. \tag{33}$$

In the coordinate system aligned with the star's spin axis, the exterior magnetic field is then described by the functions

$$\frac{1}{2} b^{1m}_0(r) = b^{1m}_+(r) = -b^{1m}_-(r) = \sqrt{\frac{4\pi}{3}} d^1_{m0}(\alpha) B_p \frac{R^3}{r^3}. \tag{34}$$

In order to compute the eigenmodes of a star with an dipole magnetic field exterior to the star, the magnetic field inside of the star must be specified. In this paper we will only consider interior fields which are dipole (i.e. $\ell = 1$). General formulae for the perturbations of a star with any interior dipole field will be presented in the following section. As examples we will consider two simple interior magnetic fields which can be matched to the dipole exterior (in the sense that the normal component of the magnetic field at the star's surface is continuous). Both of these examples are force-free fields which obey

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu \mathbf{B}, \tag{35}$$

where μ is a constant with units of inverse length. Force-free magnetic fields have the simple feature that they do not distort the star's equilibrium fluid configuration.

The first example is the simplest possible interior field, the uniform magnetic field, corresponding to $\mu = 0$ in equation (35). The coefficients appearing in equation (26) are

$$\beta_1 = -\beta_2 = B_{in}, \quad \beta_3 = 0, \tag{36}$$

where B_{in} is the magnetic field at the centre of the star. Since the radial component of the magnetic field must be continuous at the surface of the star, $B_{in} = 2B_p$. The magnetic field coefficients in the coordinate system aligned with the spin axis are

$$b^{1,m}_0(r) = b^{1,m}_-(r) = -b^{1,m}_+(r) = \sqrt{\frac{4\pi}{3}} B_{in} d^1_{m0}(\alpha). \tag{37}$$

The second example corresponds to the force-free condition (35) with μ nonzero discussed by Ferraro & Plumpton (1966). The magnetic field coefficients are

$$\beta_1(r) = 3B_{in} \frac{j_1(\mu r)}{\mu r}, \quad \beta_2(r) = -\frac{3}{2} B_{in} \frac{1}{\mu} \left(\frac{d}{dr} + \frac{1}{r} \right) j_1(\mu r), \quad \beta_3 = -\frac{3}{2} B_{in} j_1(\mu r), \tag{38}$$

where $j_1(\mu r)$ is a spherical Bessel function. Continuity of the normal component of the magnetic field at the surface of star requires that the constant μ be given by the solution of the equation

$$j_1(\mu R) = \frac{2}{3}\frac{B_p}{B_{in}}\mu R, \qquad (39)$$

which can be solved numerically once the ratio B_p/B_{in} is specified. The solution to equation (39) is shown in Figure 1, where the dimensionless product μR is plotted versus the ratio B_p/B_{in}. The force-free magnetic field then corresponds to family of fields parametrised by the ratio of magnetic field at the pole to the magnetic field at the centre of the star. This ratio of magnetic fields can have values in the range $0 < B_p/B_{in} \leq 0.5$, as shown in Figure 1.

4. Magnetically Modified r-modes

We now turn to the calculation of the modification of the r-modes due to dipole magnetic field. In the slow-rotation approximation, the r-modes of non-magnetic incompressible stars have the form

$$\omega_{\ell_A m_A} = -\frac{2m_A\Omega}{\ell_A(\ell_A + 1)} \qquad (40a)$$

$$W_{\ell_A m_A} = 0, \quad V_{\ell_A m_A} = 0 \qquad (40b)$$

$$U_{\ell_A m_A} = \delta_{\ell_A|m_A|}R^2\sqrt{\frac{2\pi(2\ell_A + 3)(\ell_A + 1)}{15\ell_A}}\left(\frac{r}{R}\right)^{\ell_A+1} \qquad (40c)$$

$$\epsilon_{\ell_A m_A} = T = \frac{1}{5}MR^2\Omega^2, \qquad (40d)$$

where the functions $W_{\ell_A m_A}, V_{\ell_A m_A}$ and $U_{\ell_A m_A}$ are defined in the expansion (23). Each r-mode solution is identified by a distinct quantum number m_A. At lowest order in angular velocity, the only value of ℓ_A allowed is $\ell_A = |m_A|$. The spin-weighted decomposition of the Ath r-mode is

$$\xi_A(x) = f_0^A(r)\; _0Y_{\ell_A m_A}\boldsymbol{e}_0 + f_+^A(r)\; _+Y_{\ell_A m_A}\boldsymbol{e}_- + f_-^A(r)\; _-Y_{\ell_A m_A}\boldsymbol{e}_+, \qquad (41)$$

where the functions $f_s^A(r)$ are given by

$$f_0^A(r) = 0, \quad f_+^A = f_-^A = R(\ell_A + 1)\sqrt{\frac{\pi(2\ell_A + 3)}{15}}\left(\frac{r}{R}\right)^{\ell_A}. \qquad (42)$$

In order to compute the r-mode frequencies in the presence of a magnetic field, we must calculate the values of the magnetic coupling coefficients appearing in equation (13). The first

coupling coefficient $\kappa_{AD}^{(1)}(r)$ depends on the perturbed magnetic field. The equations for the perturbed magnetic field were derived in the appendix for general magnetic fields and general fluid perturbations. In this section we will present the form of the perturbed magnetic field for r-modes in the presence of a dipole magnetic field. In general, the perturbed magnetic field has the expansion

$$\delta \mathbf{B}_A(r, \theta, \phi) = \sum_{\lambda,\mu} \left(\delta B_{A,0}^{\lambda,\mu}(r) \, _0Y_{\lambda\mu}^* \mathbf{e}_0 \; + \; \delta B_{A,+}^{\lambda,\mu}(r) \, _-Y_{\lambda\mu}^* \mathbf{e}_- \; + \; \delta B_{A,-}^{\lambda,\mu}(r) \, _+Y_{\lambda\mu}^* \mathbf{e}_+ \right). \tag{43}$$

In this equation, the summation is over all λ obeying the triangle inequality $\ell_A - 1 \leq \lambda \leq \ell_A + 1$ and all μ obeying $\mu = -\ell_A - m$. In the case of an r-mode and a dipole magnetic field, the "0" component appearing in equation (43) is given by

$$\delta B_{A,0}^{\lambda,\mu} = \sum_{m=-1}^{+1} C(\ell_A, 1, \lambda) \begin{pmatrix} \lambda & \ell_A & 1 \\ \mu & m_A & m \end{pmatrix} \begin{pmatrix} \lambda & \ell_A & 1 \\ 0 & -1 & 1 \end{pmatrix} \frac{1}{r} b_0^{1,m} f_+^A(r) \left(1 + (-1)^{\lambda+\ell_A}\right), \tag{44}$$

which is only nonzero when $\mu = -(m_A + m)$ and $\lambda = \ell_A$ and the constant $C(\ell_A, 1, \lambda)$ is given by

$$C(\ell_A, \ell, \lambda) = (-1)^{\ell_A + \lambda + \ell} \sqrt{\frac{(2\ell_A + 1)(2\ell + 1)(2\lambda + 1)}{4\pi}}. \tag{45}$$

The $\pm$ components appearing in (43) are

$$\delta B_{A,+}^{\lambda,\mu} = (-1)^{\lambda + \ell_A + 1} \delta B_{A,-}^{\lambda,\mu \, *} \tag{46a}$$

$$= \sum_{m=-1}^{+1} C(\ell_A, 1, \lambda) \begin{pmatrix} \lambda & \ell_A & 1 \\ \mu & m_A & m \end{pmatrix} \frac{1}{r} f_+^A(r) \times \tag{46b}$$

$$\left[\left((\ell_A - 1)b_0^{1,m}(r) - b_+^{1,m}(r)\right) \begin{pmatrix} \lambda & \ell_A & 1 \\ -1 & 1 & 0 \end{pmatrix} + \sqrt{\frac{\ell_A(\ell_A + 1)}{2}} b_+^{1,m}(r) \begin{pmatrix} \lambda & \ell_A & 1 \\ -1 & 0 & 1 \end{pmatrix} \right.$$

$$\left. - \sqrt{\frac{(\ell_A - 1)(\ell_A + 2)}{2}} b_-^{1,m}(r) \begin{pmatrix} \lambda & \ell_A & 1 \\ -1 & 2 & -1 \end{pmatrix} \right].$$

In equation (46b) λ takes on values $\ell_A, \ell_A \pm 1$. Evaluating the terms appearing in square brackets in equation (46b), the individual terms for the allowed values of λ are

$$\delta B_{A,+}^{\ell_A,\mu} = (-1)^{\ell_A + 1} \sqrt{\frac{3(2\ell_A + 1)}{4\pi \ell_A(\ell_A + 1)}} \frac{f_+^A(r)}{r} \tag{47a}$$

$$\times \sum_{m=-1}^{+1} \left[(\ell_A - 1)b_0^{1,m} - (b_+^{1,m} + b_+^{1,m*}) - \frac{1}{2}\ell_A(\ell_A + 1)(b_+^{1,m} - b_+^{1,m*}) \right] \begin{pmatrix} \ell_A & \ell_A & 1 \\ -(\ell_A + m) & \ell_A & m \end{pmatrix}$$

$$\delta B_{A,+}^{\ell_A-1,\mu} = (-1)^{\ell_A+1}\sqrt{\frac{3(\ell_A-1)(\ell_A+1)}{4\pi\ell_A}}\frac{f_+^A(r)}{r} \tag{47b}$$

$$\times \sum_{m=-1}^{+1}\left[(\ell_A-1)b_0^{1,m}-\frac{1}{2}(\ell_A+2)(b_+^{1,m}+b_+^{1,m*})\right]\begin{pmatrix}\ell_A-1 & \ell_A & 1\\ -(\ell_A+m) & \ell_A & m\end{pmatrix}$$

$$\delta B_{A,+}^{\ell_A+1,\mu} = (-1)^{\ell_A}\sqrt{\frac{3\ell_A(\ell_A+2)}{4\pi(\ell_A+1)}}\frac{f_+^A(r)}{r} \tag{47c}$$

$$\times \sum_{m=-1}^{+1}\left[(\ell_A-1)b_0^{1,m}+\frac{1}{2}(\ell_A-1)(b_+^{1,m}+b_+^{1,m*})\right]\begin{pmatrix}\ell_A+1 & \ell_A & 1\\ -(\ell_A+m) & \ell_A & m\end{pmatrix}.$$

The first coupling coefficient $\kappa_{AD}^{(1)}$ is given by equation (A13). The first summation appearing in equation (A13) reduces to

$$\sum_{\lambda,\mu}\delta B_{A,0}^{*\lambda,\mu}\delta B_{D,0}^{\lambda,\mu} = 4\delta_{AD}C^2(\ell_A,1,\ell_A)\sum_{m=-1}^{+1}\left(\frac{1}{r}b_0^{1,m}f_+^A(r)\right)^2\begin{pmatrix}\ell_A & \ell_A & 1\\ \mu & m_A & m\end{pmatrix}^2\begin{pmatrix}\ell_A & \ell_A & 1\\ 0 & -1 & 1\end{pmatrix}^2. \tag{48}$$

Making use of the known values of the Wigner 3-j symbols (Edmonds 1974), equation (48) reduces to

$$\sum_{\lambda,\mu}\delta B_{A,0}^{*\lambda,\mu}\delta B_{D,0}^{\lambda,\mu} = \frac{2}{\ell_A+1}\delta_{AD}\left(\frac{1}{r}\beta_1 f_+^A(r)\right)^2\left(\ell_A\cos^2\alpha+\frac{1}{2}\sin^2\alpha\right). \tag{49}$$

Due to the symmetry property (46a), the final two summations appearing in (A13) are

$$\sum_{\lambda,\mu}\left(\delta B_{A,+}^{*\lambda,\mu}\delta B_{D,+}^{\lambda,\mu}+\delta B_{A,-}^{*\lambda,\mu}\delta B_{D,-}^{\lambda,\mu}\right) = \sum_{\lambda,\mu}\delta B_{A,+}^{*\lambda,\mu}\delta B_{D,+}^{\lambda,\mu}+(-1)^{\ell_A+\ell_D}\delta B_{A,+}^{\lambda,\mu}\delta B_{D,+}^{*\lambda,\mu}. \tag{50}$$

Since the triangle inequalities $\ell_A-1\le\lambda\le\ell_A+1$ and $\ell_D-1\le\lambda\le\ell_D+1$ must both be satisfied, only modes satisfying $\ell_D=\ell_A,\ell_A\pm1,\ell_A\pm2$ have non-zero magnetic coupling. As a result the only nonzero entries in the coupling coefficient matric κ_{AD} will be those coupling the modes satisfying the triangle inequality.

The general equation for the diagonal elements of the coupling coefficient matrix $\kappa_{AA}^{(1)}(r)$, for any interior dipole magnetic field is

$$\kappa_{AA}^{(1)}(r) = 2\left(\frac{f_+(r)}{r}\right)^2\times\left[\frac{1}{\ell_A+1}(\beta_1)^2\left(\ell_A\cos^2\alpha+\frac{1}{2}\sin^2\alpha\right)\right. \tag{51}$$

$$\left.+\frac{1}{\ell_A(\ell_A+1)^2}\left(((\ell_A-1)\beta_1-2\beta_2)^2+(\beta_3)^2\right)\left(\ell_A\cos^2\alpha+\frac{1}{2}\sin^2\alpha\right)\right.$$

$$+\frac{(\ell_A - 1)(\ell_A + 1)}{2\ell_A(2\ell_A + 1)} \left((\ell_A - 1)\beta_1 - (\ell_A + 2)\beta_2\right)^2 \sin^2\alpha$$

$$+\frac{\ell_A(\ell_A + 2)(\ell_A - 1)^2}{(2\ell_A + 3)(\ell_A + 1)^2} (\beta_1 + \beta_2)^2 \left(\cos^2\alpha + \frac{(2\ell_A^2 + 3\ell_A + 2)}{2(2\ell_A + 1)}\right)\Bigg]$$

Similarly, the function $\kappa_{AA}^{(2)}$ has the value

$$\kappa_{AA}^{(2)}(r) = \left(\frac{f_+(r)}{r}\right)^2 \frac{1}{(\ell_A + 1)(2\ell_A + 1)} \times \tag{52}$$

$$\left(2\beta_1\left(\frac{d}{dr}(r\beta_2) + \beta_1\right) - 2(\beta_3)^2\right)\left(\ell_A \cos^2\alpha + \frac{1}{2}\sin^2\alpha\right)$$

The frequency correction can now be computed for any interior dipole magnetic field, once the functions $\beta_1(r), \beta_2(r)$ and $\beta_3(r)$ which describe the equilibrium magnetic field (see equation (26)) have been determined.

4.1.   Frequency Correction for a Uniform Magnetic Field

The simplest example of an interior magnetic field is the uniform magnetic field which has $\beta_1(r) = -\beta_2(r) = B_{in}$ and $\beta_3(r) = 0$. Since the magnetic field is uniform, it satisfies the equation $\nabla \times \mathbf{B} = 0$. As a result, the term $\kappa_{AD}^{(2)}$ vanishes. The term $\kappa_{AD}^{(3)}$ does not vanish, but its contribution to the complete coupling coefficient κ_{AD} is a factor of $\Omega^2 R^2/c^2$ smaller than the contribution from $\kappa_{AD}^{(1)}$, so we neglect it. The magnetic coupling coefficients between different r-modes can now be computed. The only nonvanishing coefficient is the self-coupling term κ_{AA} given by equation (51). A straight-forward calculation yields

$$\kappa_{AA} = -\frac{1}{5}(\ell_A + 1)(2\ell_A + 3)\left[1 + \frac{1}{2}(\ell_A(\ell_A + 1) - 3)\sin^2\alpha\right]. \tag{53}$$

Since the matrix κ is diagonal for the case of coupled r-modes, the frequencies σ_A of the magnetically modified r-modes are simply

$$\sigma_A = \omega_A\left(1 + \frac{\mathcal{M}}{2T}|\kappa_{AA}|\right). \tag{54}$$

The final equations (53) and (54) for the frequency correction for the r-modes is surprisingly simple. However, this simplicity is a result of the simplicity of the background magnetic field. Since we assume that the field is homogeneous, terms involving derivatives of the magnetic field vanish, with the net result that the matrix κ_{AD} is diagonal.

In this simple example, we have not considered couplings between r-modes and other families of modes, such as the f-modes or inertial modes. By neglecting these couplings, we are neglecting off-diagonal terms in the matrix κ_{AD}, which is only valid in the approximation that the energy in the magnetic field is small compared to the star's rotational and gravitational potential energies. For this reason equation (54) is the frequency correction to the r-modes only in the weak-field approximation. We will compute the couplings to other modes elsewhere, which will allow the calculation of the modes of rotating stars with arbitrarily large magnetic field.

4.2. Frequency Corrections for a force-free Magnetic Field

The second model for the interior magnetic field which we have examined is the force-free magnetic field, which is described by the functions $\beta_1(r), \beta_2(r)$ and $\beta_3(r)$ given by equation (36). All three functions are proportional to B_{in}, which is the magnetic field strength at the centre of the star, which is not a known quantity. Typically, only the exterior dipole magnetic field at the pole of the star, B_p, can be measured, and the interior magnetic field is unknown. When the interior field is modeled by a force-free field, the ratio of B_p/B_{in} is a free parameter. The smallest ratio of interior to exterior magnetic fields allowed in the force-free magnetic field model is $B_p/B_{in} = 0.5$, which corresponds to the uniform magnetic field (see Figure 1).

Due to the toroidal symmetry of the fluid displacement vector, $\kappa_{AA}^{(2)}$ vanishes for the force-free magnetic field so that only $\kappa_{AA}^{(1)}$ needs to be computed. This computation can be easily done numerically and the results are shown in Table 2 and Figure 2. The frequency correction depends on the strength of the internal magnetic field and the angle between the spin axis and magnetic field axis. In order to show the dependence on both of these parameters, we have written the correction term in the form $\kappa_{AA} = a_{AA} + b_{AA} \sin^2 \alpha$, where a_{AA} and b_{AA} depend only on the ratio of external to internal magnetic field B_p/B_{in}. In Table 2 we show the values of the magnetic frequency correction terms for various values of ℓ_A and for the specific value of magnetic field ratio $B_p/B_{in} = 0.1$. It can be seen that the coupling coefficients grow with increasing values of the spherical harmonic index ℓ_A. This property agrees with what is seen in the case of the uniform magnetic field.

In Figure 2 we show the values of the frequency correction for the case $\ell_A = m_A = 2$ for the complete range of allowed interior magnetic field strengths. Figure 2 shows that the magnitude of the magnetic coupling coefficients decreases as the interior magnetic field is increased, which seems counter-intuitive. However, in order to find the corrected frequency in equation (54), the coupling coefficient is multiplied by $\mathcal{M}$, the energy of the internal

magnetic field. Now $\mathcal{M} \propto B_{in}^2$, and the internal magnetic field is not known: all that we can measure is the external dipole magnetic field strength B_p. Once B_p is known, the fraction B_p/B_{in} is a free parameter of the theory. Therefore, the true scaling of the frequency correction due to magnetic fields is $(B_{in}/B_p)^2 \kappa_{AA}$. Once this is taken into account, we see that the magnitude of the frequency correction increases with increasing interior magnetic field strength, as would be expected.

5. Conclusions

In this paper we have presented a formalism for computing the oscillation modes of magnetic rotating stars. This method is based on the phase-space mode decomposition introduced by Schenk et al. (2002) and makes use of spin-weighted spherical harmonics. As a simple example, we have computed the magnetic corrections to the r-modes of an incompressible star with a uniform magnetic field in the weak-magnetic field limit. In this limit, the spatial form of the r-mode velocity field is unchanged by the presence of a magnetic field. In the case of a uniform interior magnetic field matched to an exterior dipole field, the resulting formula for the frequency has the simple form presented in equations (53) and (54). Qualitatively similar results were found for the force-free magnetic field. The presence of a magnetic field increases the absolute value of the rotating frame frequency, so that the mode will be counter-rotating *faster* than if the star had no magnetic field. Adding a magnetic field to the star increases the restoring force acting on the fluid and increases the oscillation frequency, just as increasing the spring constant of a spring causes a spring to undergo faster oscillations. Mathematically, the increase in frequency results from the fact that the diagonal entries in the matrix κ are negative. Since these diagonal entries correspond to the work done by the magnetic field on the fluid, the negative sign denotes the fact that the magnetic field opposes the fluid motion. Hence, the increase in frequency which we have found agrees with basic physical intuition.

Although we have only presented detailed results for two types of internal dipole field, the equations presented in section 4 can be used to determine the frequency correction for any other internal dipole field. Once the functions $\beta_1(r) - \beta_3(r)$ describing the equilibrium magnetic field have been found, it is only necessary to substitute them into equations (51) and (52) and perform the integration over volume in equation (13).

As long as the magnetic field energy is small compared to the star's rotational energy, the CFS gravitational-radiation-driven instability criterion Friedman & Schutz (1978b) will be unchanged. As a result, the magnetic field has a slight stabilizing effect on the r-modes, since it is now harder for the mode to be dragged forward by the star's rotation. This

agrees with the intuitive notion that magnetic fields tend to oppose fluid motion. However, in order for the mode to be stable to the CFS mechanism, it would be necessary for the magnetic field's energy to be of the same order of magnitude as the rotational energy, which corresponds to a regime where our present results do no hold. Such a large field would presumably alter the stability criterion (Friedman & Schutz 1978b) as well.

Although our calculations for the r-modes of magnetic stars is only valid in the weak-field limit, we note that a young neutron star with spin period and magnetic field similar to the Crab's will have $\mathcal{M}/T$ of the order 10^{-9} well within the regime of the weak-field limit. Since we do not include a crust, the applicability of our equations is mainly to young neutron stars with $T > 10^{10}K$ or other stars without a solid layer.

Our calculations are purely linear, so the present work does not shed any light on the nonlinear evolution of an unstable mode of a magnetic star which has been investigated by Rezzolla et al. (2000). However, the frequency change due to the magnetic field which we have calculated will alter the kinematic drift found by Rezzolla et al. (2000), although we doubt that these changes would alter their qualitative results. It would be interesting to see how the magnetically modified r-modes presented in this paper affect the evolution of spin and magnetic field in young neutron stars.

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A. Derivation of the Magnetic Coupling Coefficients for General Magnetic Fields

In this appendix we give details about the calculation of the magnetic coupling coefficients which appear in equation (13). The first coupling coefficient $\kappa_{AD}^{(1)}$ depends on the perturbed magnetic field, defined by $\delta \mathbf{B}_A = \nabla \times (\xi_A \times B)$, where B is the equilibrium magnetic field and ξ_A is the fluid displacement vector. This definition of the perturbed magnetic field can be rewritten in the form

$$\delta B_A^a = (\xi_A^a B^b - \xi_A^b B^a)_{;b}, \tag{A1}$$

where the subscripted semicolon denotes covariant differentiation, and lower case Latin subscripts and superscripts are vector indices. The components of the perturbed magnetic field in the basis (e_0, e_+, e_-) can be found by substituting the expansions (20a) and (31a) into equation (A1). The radial component of the perturbed magnetic field is given by

$$\begin{aligned}
\delta B_A^0 &= \delta B_A \cdot e_0 = \xi_{A\ ;b}^0 B^b + \xi_A^0 B_{;b}^b - \xi_{A\ ;b}^b B^0 - \xi_A^b B_{;b}^0 \\
&= \xi_{A\ ;+}^0 B^+ + \xi_{A\ ;-}^0 B^- + \xi_A^0 B_{;+}^+ + \xi_A^0 B_{;-}^- - \xi_{A\ ;+}^+ B^0 - \xi_{A\ ;-}^- B^0 - \xi_A^+ B_{;+}^0 - \xi_A^- B_{;-}^0
\end{aligned} \tag{A2}$$

Note that although it is true that $B^b_{;b} = 0$, we gain an equation with a more symmetric form by not explicitly setting the divergence of the magnetic field to zero. Since each component of the equilibrium magnetic field and the fluid displacement vector is proportional to a spin-spherical harmonic, the perturbed magnetic field components can be written in the compact form

$$\delta B_{A,s}(r, \theta, \phi) = \sum_{\ell_A} \sum_{\ell,m} \sum_{s_A+s=s} \zeta^{A\Lambda}_{s,s_A,s}(r) \, {}_{s_A}Y_{\ell_A m_A} \, {}_sY_{\ell m} \tag{A3}$$

where s takes on values of 0 and ± 1, and the radial functions $\zeta^{A\Lambda}_{s,s_A,s}(r)$ will be determined shortly. In equation (A3), the integers ℓ_A, m_A, s_A are the fluid perturbation's quantum numbers, ℓ, m, s are the equilibrium magnetic field's quantum numbers and the symbol Λ represents the set of quantum numbers (ℓ, m) describing the magnetic field.

A detailed formalism for taking covariant derivatives of vectors using spherical coordinates has been given in section VI of Schenk et al. (2002). Their equations (6.15) and (6.16) summarise the rules for taking covariant derivatives in the spin-weighted basis. (The only difference in notation between the present paper and that of Schenk et al. (2002) is the symbols used for the basis vectors. Schenk et al. (2002) use the basis $\{\mathbf{l}, \mathbf{m}, \bar{\mathbf{m}}\}$ which are related to our basis by $\mathbf{l} = \boldsymbol{e}_0$, $\mathbf{m} = \boldsymbol{e}_+$ and $\bar{\mathbf{m}} = \boldsymbol{e}_-$.) When these covariant differentiation rules are used, we find that the $s = 0$ terms of $\zeta^{A\Lambda}$ appearing in (A3) are

$$\zeta^{A\Lambda}_{0,0,0}(r) = \sqrt{\frac{\ell(\ell+1)}{2}} \frac{f^A_0}{r}(b^\Lambda_+ - b^\Lambda_-) - \sqrt{\frac{\ell_A(\ell_A+1)}{2}} \frac{b^\Lambda_0}{r}(f^A_+ - f^A_-) \tag{A4a}$$

$$\zeta^{A\Lambda}_{0,1,-1}(r) = -\left(\sqrt{\frac{\ell(\ell+1)}{2}} \frac{f^A_+ b^\Lambda_0}{r} + \sqrt{\frac{\ell_A(\ell_A+1)}{2}} \frac{f^A_0 b^\Lambda_-}{r} \right) \tag{A4b}$$

$$\zeta^{A\Lambda}_{0,-1,1}(r) = \sqrt{\frac{\ell(\ell+1)}{2}} \frac{f^A_- b^\Lambda_0}{r} + \sqrt{\frac{\ell_A(\ell_A+1)}{2}} \frac{f^A_0 b^\Lambda_+}{r}. \tag{A4c}$$

Similar manipulations give us the $\pm$ components of the perturbed magnetic field. In particular, the $+$ component of δB is

$$\begin{aligned} \delta B_+ = \delta B^- = \delta B \cdot \boldsymbol{e}_+ = \quad & \xi^-_{;b}B^b + \xi^- B^b_{;b} - \xi^b_{;b}B^- - \xi^b B^-_{;b} \\ = \quad & \xi^-_{;+}B^+ + \xi^-_{;0}B^0 + \xi^- B^+_{;+} + \xi^- B^0_{;0} \\ & -\xi^+_{;+}B^- - \xi^0_{;0}B^- - \xi^+ B^-_{;+} - \xi^0 B^-_{;0}. \end{aligned} \tag{A5}$$

Substitution of the covariant differentiation rules into (A5) and comparing with equation (A3) results in

$$\zeta^{A\Lambda}_{1,1,0}(r) = \frac{1}{r}\frac{d}{dr}\left(r b^\Lambda_0 f^A_+\right) - \sqrt{\frac{\ell(\ell+1)}{2}} \frac{b^\Lambda_- f^A_+}{r} \tag{A6a}$$

$$\zeta_{1,0,1}^{A\Lambda}(r) \;=\; -\frac{1}{r}\frac{d}{dr}\left(rb_+^{\Lambda}f_0^A\right) + \sqrt{\frac{\ell_A(\ell_A+1)}{2}}\,\frac{b_+^{\Lambda}f_-^A}{r} \tag{A6b}$$

$$\zeta_{1,2,-1}^{A\Lambda}(r) \;=\; -\sqrt{\frac{(\ell_A-1)(\ell_A+2)}{2}}\,\frac{b_-^{\Lambda}f_+^A}{r} \tag{A6c}$$

$$\zeta_{1,-1,2}^{A\Lambda}(r) \;=\; \sqrt{\frac{(\ell-1)(\ell+2)}{2}}\,\frac{b_+^{\Lambda}f_-^A}{r}. \tag{A6d}$$

In the case of a purely dipole magnetic field, $\ell = 1$ and the term $\zeta_{1,-1,2}^{A\Lambda}(r)$ vanishes. Similarly, the $s = -1$ components of ζ are

$$\zeta_{-1,-1,0}^{A\Lambda}(r) \;=\; \frac{1}{r}\frac{d}{dr}\left(rb_0^{\Lambda}f_-^A\right) + \sqrt{\frac{\ell(\ell+1)}{2}}\,\frac{b_+^{\Lambda}f_-^A}{r} \tag{A7a}$$

$$\zeta_{-1,0,-1}^{A\Lambda}(r) \;=\; -\frac{1}{r}\frac{d}{dr}\left(rb_-^{\Lambda}f_0^A\right) - \sqrt{\frac{\ell_A(\ell_A+1)}{2}}\,\frac{b_-^{\Lambda}f_+^A}{r} \tag{A7b}$$

$$\zeta_{-1,-2,1}^{A\Lambda}(r) \;=\; \sqrt{\frac{(\ell_A-1)(\ell_A+2)}{2}}\,\frac{b_+^{\Lambda}f_-^A}{r} \tag{A7c}$$

$$\zeta_{-1,1,-2}^{A\Lambda}(r) \;=\; -\sqrt{\frac{(\ell-1)(\ell+2)}{2}}\,\frac{b_-^{\Lambda}f_+^A}{r}. \tag{A7d}$$

In equation (A3), the components of perturbed magnetic field were written as products of spin-spherical harmonics. This equation can be simplified so that each term is proportional to only one spin-spherical harmonic,

$$\delta\mathbf{B}_{A,s}(r,\theta,\phi) = \sum_{\lambda,\mu} \delta B_{A,s}^{\lambda,\mu}(r)\; {}_sY_{\lambda\mu}^*, \tag{A8}$$

by using the combination formula

$${}_sY_{\ell m}(\theta,\phi)\, {}_tY_{kn}(\theta,\phi) = \sum_{\lambda,\mu,\sigma} \sqrt{\frac{(2\ell+1)(2k+1)(2\lambda+1)}{4\pi}} \begin{pmatrix} \ell & k & \lambda \\ -s & -t & -\sigma \end{pmatrix} \begin{pmatrix} \ell & k & \lambda \\ m & n & \mu \end{pmatrix} {}_\sigma Y_{\lambda\mu}^*(\theta,\phi) \tag{A9}$$

where the summation is over all values of λ satisfying the triangle inequality $|\ell - k| \le \lambda \le \ell + k$, μ satisfying $\mu = -(m+n)$, σ satisfying $\sigma = -(s+t)$ and $\begin{pmatrix} \ell & k & \lambda \\ -s & -t & -\sigma \end{pmatrix}$ is a Wigner 3-j symbol (Edmonds 1974). The radial expansion functions $\delta B_{A,s}^{\lambda,\mu}(r)$ are given by

$$\delta B_{A,s}^{\lambda,\mu}(r) = \sum_{m=-1}^{+1} C(\ell_A,1,\lambda) \begin{pmatrix} \lambda & \ell_A & 1 \\ \mu & m_A & m \end{pmatrix} \sum_{s_A+s=s} \begin{pmatrix} \lambda & \ell_A & 1 \\ -s & s_A & s \end{pmatrix} \zeta_{s,s_A,s}^{A\Lambda}(r), \tag{A10}$$

and the constant $C(\ell_A, \ell, \lambda)$ is defined by

$$C(\ell_A, \ell, \lambda) = (-1)^{\ell_A + \lambda + \ell} \sqrt{\frac{(2\ell_A + 1)(2\ell + 1)(2\lambda + 1)}{4\pi}}. \tag{A11}$$

Since the spin-spherical harmonics obey the orthogonality relation

$$\int_{4\pi} {}_sY^*_{\ell'm'} \, {}_sY_{\ell m} d\Omega = \delta_{\ell\ell'}\delta_{mm'}, \tag{A12}$$

the first magnetic coupling coefficient is just the summation

$$\kappa^{(1)}_{AD}(r) = \int \delta\mathbf{B}^*_A \cdot \delta\mathbf{B}_D d\Omega = \sum_{\lambda,\mu} \left(\delta B^{*\lambda,\mu}_{A,0} \delta B^{\lambda,\mu}_{D,0} + \delta B^{*\lambda,\mu}_{A,+} \delta B^{\lambda,\mu}_{D,+} + \delta B^{*\lambda,\mu}_{A,-} \delta B^{\lambda,\mu}_{D,-} \right). \tag{A13}$$

Similar expressions for the coupling coefficients $\kappa^{(2)}_{AD}(r)$ and $\kappa^{(3)}_{AD}(r)$ can also be written. In order to calculate $\kappa^{(2)}_{AD}(r)$, the vector $\boldsymbol{\xi} \times (\boldsymbol{\nabla} \times \mathbf{B})$ needs to be decomposed into a sum over the spin-weighted spherical harmonics. The decomposition is of the form

$$\boldsymbol{\chi}_A = \boldsymbol{\xi}_A \times (\boldsymbol{\nabla} \times \mathbf{B}) = \sum_{\lambda,\mu} \left(\chi^{\lambda,\mu}_{A,0}(r) \, {}_0Y^*_{\lambda\mu}\boldsymbol{e}_0 \; + \; \chi^{\lambda,\mu}_{A,+}(r) \, {}_-Y^*_{\lambda\mu}\boldsymbol{e}_- \; + \; \chi^{\lambda,\mu}_{A,-}(r) \, {}_+Y^*_{\lambda\mu}\boldsymbol{e}_+ \right), \tag{A14}$$

where the coefficients are

$$\chi^{\lambda,\mu}_{A,s}(r) = \sum_{\ell_A}\sum_{\ell,m} C(\ell_A, \ell, \lambda) \begin{pmatrix} \lambda & \ell_A & \ell \\ \mu & m_A & m \end{pmatrix} \sum_{s_A+s=s} \begin{pmatrix} \lambda & \ell_A & \ell \\ -s & s_A & s \end{pmatrix} \chi^{A\Lambda}_{s,s_A,s}(r). \tag{A15}$$

The only non-zero functions $\chi^{A\Lambda}(r)$ are given by the expressions

$$\chi^{A\Lambda}_{0,-1,1}(r) = \frac{1}{r}f^A_-(b^\Lambda_0 + (rb^\Lambda_+)_{,r}), \qquad \chi^{A\Lambda}_{0,1,-1}(r) = -\frac{1}{r}f^A_+(b^\Lambda_0 - (rb^\Lambda_-)_{,r}) \tag{A16a}$$

$$\chi^{A\Lambda}_{1,0,1}(r) = -\frac{1}{r}f^A_0(b^\Lambda_0 + (rb^\Lambda_+)_{,r}), \qquad \chi^{A\Lambda}_{1,1,0}(r) = -\frac{1}{r}f^A_+(b^\Lambda_+ + b^\Lambda_-) \tag{A16b}$$

$$\chi^{A\Lambda}_{-1,0,-1}(r) = \frac{1}{r}f^A_0(b^\Lambda_0 - (rb^\Lambda_-)_{,r}), \qquad \chi^{A\Lambda}_{-1,-1,0}(r) = \frac{1}{r}f^A_-(b^\Lambda_+ + b^\Lambda_-). \tag{A16c}$$

The second coupling coefficient $\kappa^{(2)}_{AD}$ is now given by

$$\kappa^{(2)}_{AD}(r) = \int \boldsymbol{\chi}^*_A \cdot \delta\mathbf{B}_D d\Omega = \sum_{\lambda,\mu} \left(\chi^{*\lambda,\mu}_{A,0} \delta B^{\lambda,\mu}_{D,0} + \chi^{*\lambda,\mu}_{A,+} \delta B^{\lambda,\mu}_{D,+} + \chi^{*\lambda,\mu}_{A,-} \delta B^{\lambda,\mu}_{D,-} \right). \tag{A17}$$

Similarly, we define the vector

$$\boldsymbol{\gamma}_A = \boldsymbol{\xi}_A \times \mathbf{B} = \sum_{\lambda,\mu} \left(\gamma^{\lambda,\mu}_{A,0}(r) \, {}_0Y^*_{\lambda\mu}\boldsymbol{e}_0 \; + \; \gamma^{\lambda,\mu}_{A,+}(r) \, {}_-Y^*_{\lambda\mu}\boldsymbol{e}_- \; + \; \gamma^{\lambda,\mu}_{A,-}(r) \, {}_+Y^*_{\lambda\mu}\boldsymbol{e}_+ \right), \tag{A18}$$

where the expansion coefficients are

$$\gamma_{A,s}^{\lambda,\mu}(r) = \sum_{\ell_A}\sum_{\ell,m} C(\ell_A,\ell,\lambda)\begin{pmatrix}\lambda & \ell_A & \ell \\ \mu & m_A & m\end{pmatrix}\sum_{s_A+s=s}\begin{pmatrix}\lambda & \ell_A & \ell \\ -s & s_A & s\end{pmatrix}\gamma_{s,s_A,s}^{A\Lambda}(r). \qquad (A19)$$

The non-zero coefficients $\gamma^{A\Lambda}(r)$ have the values

$$\gamma_{0,-1,1}^{A\Lambda}(r) = if_-^A b_+^\Lambda , \qquad \gamma_{0,1,-1}^{A\Lambda}(r) = -if_+^A b_-^\Lambda \qquad (A20a)$$
$$\gamma_{1,1,0}^{A\Lambda}(r) = if_+^A b_0^\Lambda , \qquad \gamma_{1,0,1}^{A\Lambda}(r) = -if_0^A b_+^\Lambda \qquad (A20b)$$
$$\gamma_{-1,-1,0}^{A\Lambda}(r) = -if_-^A b_0^\Lambda , \qquad \gamma_{-1,0,-1}^{A\Lambda}(r) = -if_0^A b_-^\Lambda. \qquad (A20c)$$

The third coupling coefficient is

$$\kappa_{AD}^{(3)}(r) = \int \boldsymbol{\gamma}_A^* \cdot \boldsymbol{\gamma}_D d\Omega = \sum_{\lambda,\mu}\left(\gamma_{A,0}^{*\lambda,\mu}\gamma_{D,0}^{\lambda,\mu} + \gamma_{A,+}^{*\lambda,\mu}\gamma_{D,+}^{\lambda,\mu} + \gamma_{A,-}^{*\lambda,\mu}\gamma_{D,-}^{\lambda,\mu}\right). \qquad (A21)$$

Table 1. Typical ratios of magnetic field energy to gravitational and rotational energies for different types of stars, assuming uniform density and magnetic field. The energy in the magnetic field is $\mathcal{M} = B_{in}^2 R^3/6$, the gravitational potential energy is $|W| = 3GM^2/5R$ and the rotational energy is $T = MR^2(2\pi/P)^2/5$. White dwarf data is from Wickramasinghe & Ferrario (2000), Ro AP star data is from Kurtz (1990), and the data for SGR 0526-66 is from Duncan (1998).

Name	Type	B (G)	Period (s)	Mass $(M_\odot)$	Radius (cm)	$\mathcal{M}/|W|$	$\mathcal{M}/T$
Crab	NS	1.0e+12	3.3e-02	1.4	1.0e+06	5.4e-13	8.3e-09
AP stars	RoAp	1.0e+03	1.0e+06	2.0	1.4e+11	1.0e-10	7.4e-07
AM Her	WD	1.3e+07	1.1e+04	0.7	1.0e+09	3.6e-10	3.2e-04
PG 1015+14	WD	1.6e+08	5.9e+03	0.7	1.0e+09	5.5e-08	1.4e-02
PG 1031+234	WD	1.0e+09	1.2e+04	0.7	1.0e+09	2.1e-06	2.3e+00
SGR 0526-66	NS	1.0e+15	8.0e+00	1.4	1.0e+06	5.4e-07	4.9e+02

Table 2. The numerical values of κ_{AA} for force-free magnetic field. The data are calculated for $\mu = 3.51835$ and $\frac{B_P}{B_{in}} = 0.1$. The coupling coefficients are written in the form $\kappa_{AA} = a_{AA} + b_{AA}\sin^2\alpha$ where α is the angle between the star's spin axis and the magnetic field's symmetry axis.

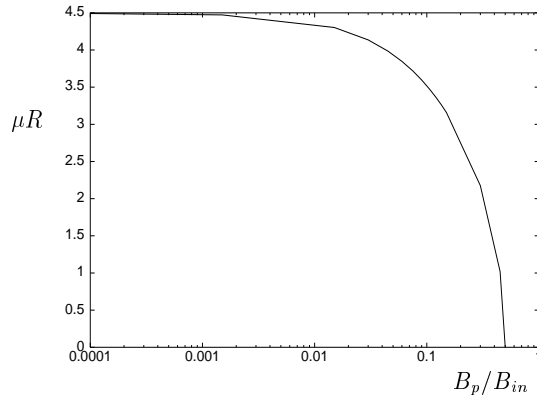| ℓ_A	a_{AA}	b_{AA}
1	-4.2182e-01	2.1091e-01
2	-5.9812e-01	2.5302e-02
3	-1.5734e+00	7.8447e-03
4	-4.1159e+00	2.9866e-01
5	-9.2792e+00	1.0233e+00
6	-1.8367e+01	2.2962e+00
7	-3.2933e+01	4.2251e+00
8	-5.4780e+01	6.9128e+00
9	-8.5963e+01	1.0458e+01
10	-1.288e+02	1.4955e+01

Fig. 1.— Plot of μR versus B_p/B_{in}, for the force-free magnetic field. In this plot R is the radius of the star, μ is the proportionality constant appearing in equation (35) and B_p/B_{in} is the ratio of external to internal magnetic fields. This plot was made by numerically finding the solution of equation (39).
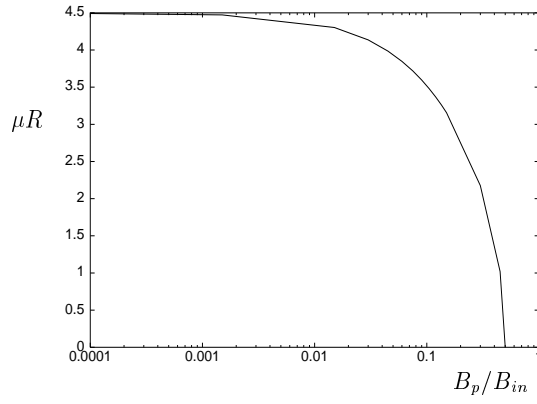
Fig. 2.— The magnetic frequency correction for an $\ell_A = m_A = 2$ r-mode as a function of B_p/B_{in}, for the force-free magnetic field. The frequency correction is written in the form $\kappa_{AA} = a_{AA} + b_{AA} \sin^2 \alpha$, and the constants a_{22} and b_{22} are plotted. The solid curve is the graph of the function a_{22} and the dot-dashed line is the graph of b_{22}.